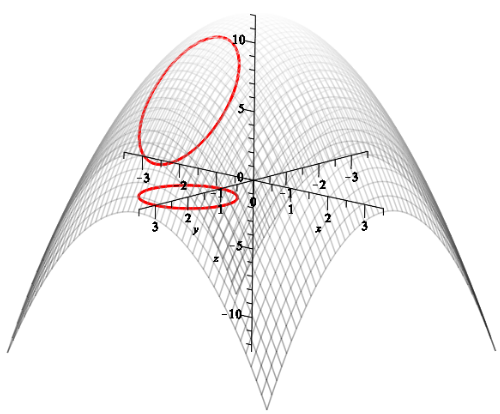
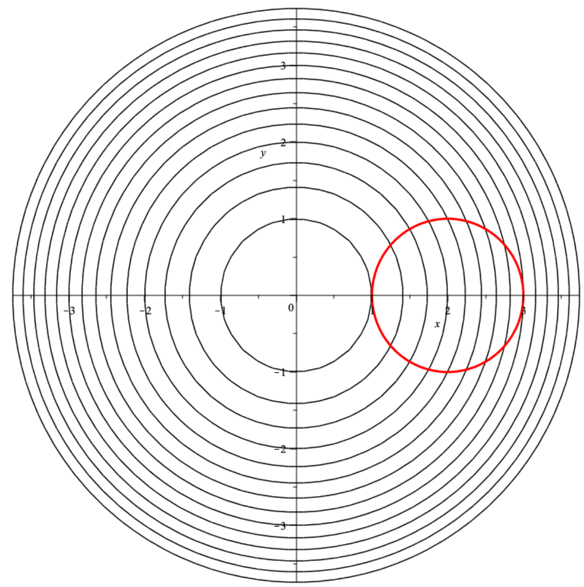
SECTION 15.8: THE METHOD OF LAGRANGE MULTIPLIERS

In the last section, we optimized functions $z = f(x, y)$ along boundary curves usually using substitution. In this method, we take advantage of the geometry of the situation more and develop the **METHOD OF LAGRANGE MULTIPLIERS**. Suppose we wish to optimize a function $z = f(x, y)$ subject to a constraint curve $g(x, y) = c$.



Constraint curve $g(x, y) = c$ (red) in the plane and lift to the surface $z = f(x, y)$.



Constraint curve $g(x, y) = c$ (red) in the plane overlaid with a contour map of $z = f(x, y)$

Note that the maximum and minimum values of $f(x, y)$ subject to $g(x, y) = c$ correspond to the highest and lowest points on the lift of $g(x, y) = c$ to the surface $z = f(x, y)$. These values, in turn, correspond to the contours of $z = f(x, y)$ which share a common tangent to the curve $g(x, y) = c$. Since $\nabla f(x, y)$ is orthogonal to the contours of f and $\nabla g(x, y)$ is orthogonal to the curve $g(x.y) = c$, we must have that $\nabla f(x, y)$ is parallel to $\nabla g(x, y)$ at the points of common tangency (i.e., the maximum and minimum.) This is Lagrange's Theorem.

THEOREM: Suppose f and g are differentiable and that f has an extreme value on $g(x, y) = c$ at the point (a, b). Then there is a real number λ called the **Lagrange Multiplier** so that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

PROOF: Let $\vec{r}(t)$ be a smooth parametrization of $g(x, y)$ so that $t = t_0$ corresponds to (a, b).

We know from earlier that $\nabla g(a, b)$ is orthogonal to $\vec{r}\,'(t_0)$. (Why?)

We also have that courtesy of the parametrization, $f(x, y) = f(t)$ (Why?)

Since $f(a, b)$ is an extreme value, and f, g, and r are differentiable, we must have that $f\,'(t_0) = 0$. (Why?)

Using the chain rule, we have that $f'(t) = f_x(x, y)\dfrac{dx}{dt} + f_y(x, y)\dfrac{dy}{dt} = \nabla f(x, y) \cdot \vec{r}\,'(t)$.

So $f'(t_0) = 0$ means $\nabla f(a, b) \cdot \vec{r}\,'(t_0) = 0$. Hence, $\nabla f(a, b)$ is also orthogonal to $\vec{r}\,'(t_0)$.

Hence, $\nabla f(a, b)$ is parallel to $\nabla g(a, b)$, so there is a real number λ so that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

THE METHOD OF LAGRANGE MULTIPLIERS

1. First establish that $f(x, y)$ has a maximum (or minimum, or both) subject to $g(x, y) = c$.
2. Next, set up the system of equations equivalent to $\nabla f(x, y) = \lambda \nabla g(x, y)$. That is:

$$\begin{cases} f_x(x, y) &= \lambda g_x(x, y) \\ f_y(x, y) &= \lambda g_y(x, y) \\ g(x, y) &= c \end{cases}$$

3. The location of the maximum (or minimum, or both) live among the solutions (x, y) to your system.

In other words, the Method of Lagrange Multipliers is a method to help us find **critical points**. It is up to us to decide (prove) whether these critical points correspond to absolute extrema.

EXAMPLE 1: Optimize $f(x, y) = 12 - x^2 - y^2$ subject to $x^2 - 4x + y^2 + 3 = 0$.

HINT: Recall that 'optimize' means find both the maximum and the minimum.

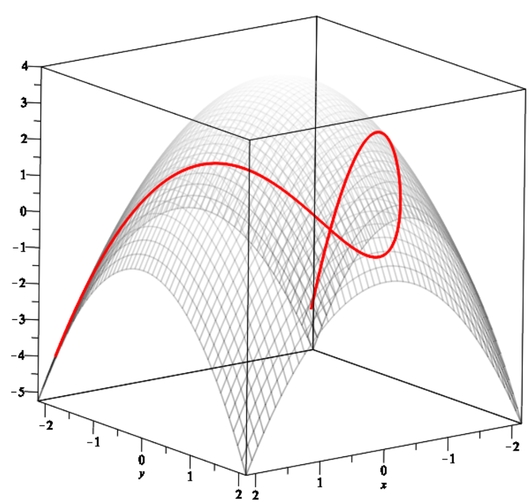
Ans: Maximum: 11 at $(1, 0)$; Minimum: 3 at $(3, 0)$.

NOTE: This scenario is exactly the one whose graphs were used to motivate the method.

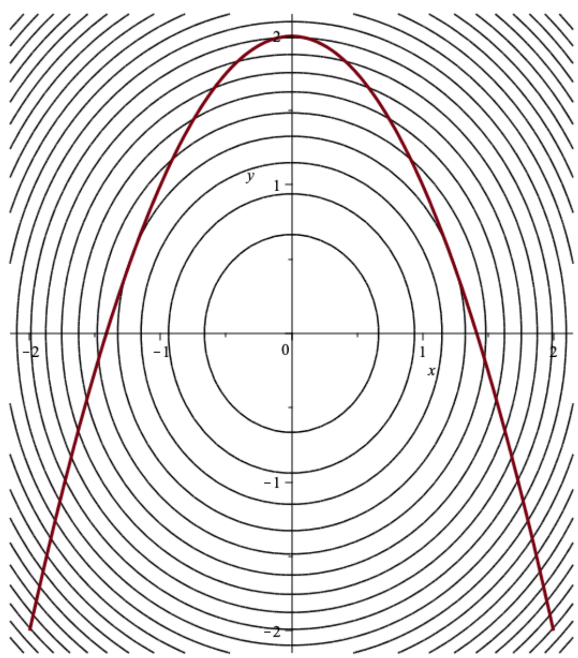
EXAMPLE 2: Maximize $z = 4 - x^2 - y^2$ subject to $y = 2 - x^2$. Why is there no mimimum?

HINT: Before you get started, rewrite the constraint in the form $g(x, y) = c$.

Ans: Maximum $\dfrac{9}{4}$ at $\left(\pm\sqrt{\dfrac{3}{2}}, \dfrac{1}{2}\right)$



The surface $f(x, y) = 4 - x^2 - y^2$ along with the lift of $y = 2 - x^2$.



The graph of $y = 2 - x^2$ overlaid with a contour map of $f(x, y) = 4 - x^2 - y^2$.

EXAMPLE 3: Use the Method of Lagrange Multipliers to show that among all rectangles of a fixed perimeter c, the one which maximizes area is a square.

HINT: Suppose the dimensions of the rectangle are x by y. Then you need to maximize $A(x, y) = xy$ subject to the constraint $2x + 2y = P$ where P is some fixed constant. How do you know what you've found is a maximum?

Ans: Maximum $\frac{P^2}{16}$ at $\left(\frac{P}{4}, \frac{P}{4}\right)$. Since $x = y = \frac{P}{4}$, the rectangle is a square.

EXTENSIONS TO MORE THAN TWO VARIABLES

EXAMPLE 4: Minimize $F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ subject to $x + 2y + 3z = 6$ and interpret geometrically.

HINT: Given a point (x, y, z), the quantity $\sqrt{x^2 + y^2 + z^2}$ computes the distance from (x, y, z) to the origin.

Ans: Minimum: $\dfrac{3\sqrt{2}}{\sqrt{7}}$ at $\left(\dfrac{3}{7}, \dfrac{6}{7}, \dfrac{9}{7}\right)$. This is the point on the plane $x + 2y + 3z = 6$ closest to the origin.

EXTENSIONS TO MORE THAN ONE CONSTRAINT

THEOREM: Suppose F, g, and h are differentiable and that F has an extreme value subject to two constraints $g(x, y, z) = d$ and $h(x, y, z) = e$ at the point (a, b, c). Then there are real numbers λ and μ so that

$$\nabla F(a, b, c) = \lambda \nabla g(a, b, c) + \mu \nabla h(a, b, c)$$

EXAMPLE 5: Optimize $F(x, y, z) = x$ subject to $z = 4 - x^2 - y^2$ and $z = y$. Interpret your answer geometrically.

Ans: Maximum: $\dfrac{\sqrt{17}}{2}$ at $\left(\dfrac{\sqrt{17}}{2}, -\dfrac{1}{2}, -\dfrac{1}{2}\right)$; Minimum: $-\dfrac{\sqrt{17}}{2}$ at $\left(-\dfrac{\sqrt{17}}{2}, -\dfrac{1}{2}, -\dfrac{1}{2}\right)$.

These are the points furthest out and back on the curve of intersection of $z = 4 - x^2 - y^2$ and $z = y$.

WHAT DOES THE LAGRANGE MULTIPLIER MEAN?

Suppose we're optimizing $f(x,y)$ subject to $g(x,y) = c$. Consider the function

$$F(x,y,\lambda) = f(x,y) + \lambda(c - g(x,y))$$

1. Show that solving:

$$\begin{cases} f_x(x,y) & = & \lambda\, g_x(x,y) \\ f_y(x,y) & = & \lambda\, g_y(x,y) \\ g(x,y) & = & c \end{cases}$$

   is equivalent to solving the system $\nabla F(x,y,\lambda) = 0$. What are we looking for by solving $\nabla F(x,y,\lambda) = 0$?

2. Show that $\dfrac{\partial F}{\partial c} = \lambda$ and interpret what this means in terms of a rate of change.

EXAMPLE 6: Interpret the λ you found with the area optimization problem earlier in this packet.

HOMEWORK: Section 15.8: 9 - 57 every other odd.